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Direct Verification of Parametric Solution for Vibration Reduction Control Problems

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For the optimal control problems formulated with limited actuator authority, the optimal solution is given by Pontryagin's minimum principle that provides the necessary and sufficient conditions of optimality for systems that are normal. If the resulting solution is a switched control, an alternate computation methodology exists where the problem is formulated as a parametric optimization problem with the switching times as the variables. For vibration reduction problems with constrained endpoints, the parametric formulation is not convex and the Karush–Kuhn–Tucker conditions can only guarantee the local optimality of the solution. In this paper an approach is presented to verify the optimality of the parametric solution for optimal control problems with terminal state inequalities. The verification conditions are derived using the switching function, Karush–Kuhn–Tucker, and the transversality equations. The resulting problem is formulated as a linear program that provides a very efficient test of optimality. Example problems are given to demonstrate the application of this algorithm.

I. Introduction

THE problem of reducing vibrations in the rest-to-rest motion of flexible structures has been the subject of active research [1–6]. The control is required to cause a finite rigid-body motion while reducing the vibrations of the flexible modes [3]. The design problem is formulated as an optimal control problem where the desired performance criteria are either included in the cost function [7] or are specified as constraints on the control [4,8,9] or the state vectors [10]. These criteria include time [3], robustness [5], fuel cost [11], power [12], deflection [13], move vibrations [10], jerk [12], higher mode excitations [14], etc. For a class of objective functions, the minimum cost formulation has been shown to be equivalent to the constrained time optimal formulation [15]. As a result, only one of the control formulations is required to be analyzed.

The optimal solution is derived using Pontryagin's minimum principle and Euler–Lagrange necessary conditions [16]. If the system is normal in the terminology of Lasalle, the minimum principle provides both the necessary as well as sufficient conditions for optimality [17]. For many of the above mentioned cost functions, the optimal control problem has been shown to be free of any singular arc. As a result, the optimal solution is either a bang–bang [3,5] or a bang–off–bang [11] switched control depending upon the switching function. The computation process requires solving complicated necessary and boundary conditions for optimality [18]. For control problems with switched control as the optimal solution, an alternative approach is to formulate a parametric optimization problem where the objective cost and the design constraints are specified analytically in terms of switching times, which are the

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variable parameters [1,19]. Efficient algorithms exist for solving the resulting nonlinear parametric optimization problem [9].

Because most of the design constraints are not convex functions of the switching times, the Karush–Kuhn–Tucker (KKT) necessary conditions cannot guarantee the global optimality of the parametric solution [20]. Additional conditions are therefore required to verify the optimality of the solution. An approach was presented by Ben-Asher [21] for the time optimal problems with given initial and final states where the zero crossings of the switching function were used to verify the switch timings. The idea has been applied to problems with higher-order flexible modes [22] and specified fuel constraints [23,24]. In this approach, the verification procedure involves finding a null space vector corresponding to the initial value of the costate vector that satisfies the zero crossing and Hamiltonian criteria. The system dynamics, along with the costates, are then used to generate the switching profile which is then compared with the parametric solution [21]. If the zero crossings in the switching profile occur only at the time instants corresponding to the switching times of the parametric solution, the solution is the true-optimal solution for the problem.

The complexity of such an approach depends upon the number of null space vectors that are required to be checked for generating the desired switching profile. The dimension of the null space in turn depends upon the relative number of control switches in the parametric solution and the order of the system. For a given problem formulation, a large number of null space vectors might need to be checked for verification of a single parametric solution. Furthermore, this approach has been only applied on problems with fixed initial and final conditions [22,23]. The problems with terminal state inequality constraints have not been studied before. These constraints appear in the vibration reduction problems if the control is desired to have specified robustness or insensitivity [25], or specified higher mode excitations [14]. For such problems, the terminal states corresponding to the additional flexible modes are restricted by inequality constraints. Because the residual vibrations are only required to be below some specified limits for these modes, the desired performance can be achieved with relatively fewer number of switchings in the solution. As a result, the dimension of the null space satisfying the zero-crossing switching criteria is very large because of the large difference in the number of states included in the model and the number of switchings in the parametric solution. Because any linear combination of the null space vectors also lies in the null space, the application of the previously reported verification technique is not practicable for such problems.

We present an efficient approach to verify the optimality of switched control for problems with terminal state inequality constraints. The conditions corresponding to the inequalities are derived from the transversality and the KKT conditions. Further, a scheme is presented where additional conditions are obtained from the switching function such that the verification and comparison can be done together during the computation of the null space vector. The resulting problem is formulated as a linear program (LP) that provides a direct test for verifying the optimality of the parametric solution. Because a LP is a convex problem [20], it can be solved very efficiently in polynomial time [26]. The analysis is presented for both bang–bang and bang–off–bang solutions.

A generalized vibration reduction problem is first formulated as an optimal control problem with terminal state inequality constraints and a specified fuel cost limit. This is based on compilation of various existing formulations presented in the literature for different types of vibration reduction problems [1–6,15,24]. Euler–Lagrange necessary conditions of optimality are then obtained from Pontryagin's minimum principle. For inequality constraints, the corresponding transversality and KKT conditions are derived. The resulting optimal control is related to the switching functions for both bang–bang and bang–off–bang profiles. For switched controls, the parametric optimization formulation is then given that is computationally more efficient. KKT necessary conditions for local optimality of the solution are provided. A new approach is then presented for the verification of the parametric solution. The sign of the switching function is used for deriving additional inequality conditions that allow direct verification of the solution. The verification problem is then formulated as a LP feasibility study problem that can be solved very efficiently. LP formulations are given both for bang–bang and bang–off–bang controls. The approach is applied on an example problem where the control is designed for specified excitation and specified fuel cost constraints. For the digital implementation, the vibration reduction problem is formulated in the discrete time domain as a convex optimization problem that directly provides the sufficient conditions for optimality.

II. Optimal Control Problem

We consider the linear flexible structures that can be decoupled using the modal decomposition technique as linear time invariant (LTI) composed of a finite number of rigid-body and flexible modes. The dynamics of such systems can be represented by the equations [3]

$$\begin{bmatrix} \dot{\boldsymbol{x}}_r(t) \\ \dot{\boldsymbol{x}}_f(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_r & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A}_f \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_r(t) \\ \boldsymbol{x}_f(t) \end{bmatrix} + \begin{bmatrix} \boldsymbol{b}_r \\ \boldsymbol{b}_f \end{bmatrix} u(t) \tag{1}$$

where $\{\boldsymbol{x}_r \in \mathbb{R}^r, \boldsymbol{A}_r, \boldsymbol{b}_r\}$ and $\{\boldsymbol{x}_f \in \mathbb{R}^f, \boldsymbol{A}_f, \boldsymbol{b}_f\}$ correspond to the rigid-body modes and the flexible modes, respectively. A simple model for such systems is that of two masses connected by a spring and viscous damper, as shown in Fig. 1. The equation of motion for such systems is given by

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t) + \boldsymbol{C}\dot{\boldsymbol{y}}(t) + \boldsymbol{K}\boldsymbol{y}(t) = \boldsymbol{D}u(t) \tag{2}$$

where $\boldsymbol{M}$ is positive definite, and $\boldsymbol{K}$ and $\boldsymbol{C}$ are positive semidefinite matrices. The linear time-invariant system can be represented as Eq. (1) by a suitable selection of parameters resulting in

$$\underbrace{\begin{bmatrix} \dot{x}_{r1} \\ \dot{x}_{r2} \end{bmatrix}}_{\dot{\boldsymbol{x}}_r} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{A}_r} \underbrace{\begin{bmatrix} x_{r1} \\ x_{r2} \end{bmatrix}}_{\boldsymbol{x}_r} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{b}_r} u \tag{3}$$

as the rigid-body mode and

$$\underbrace{\begin{bmatrix} \dot{x}_{f1} \\ \dot{x}_{f2} \end{bmatrix}}_{\dot{\boldsymbol{x}}_f} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta_1\omega_1 \end{bmatrix}}_{\boldsymbol{A}_f} \underbrace{\begin{bmatrix} x_{f1} \\ x_{f2} \end{bmatrix}}_{\boldsymbol{x}_f} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\boldsymbol{b}_f} u \tag{4}$$

as a second-order flexible mode where the mode parameters $\{\omega_1, \zeta_1\}$ can be computed from the $\boldsymbol{M}$, $\boldsymbol{K}$, and $\boldsymbol{C}$ matrices [3].

For the vibration reduction problems, the control is required to move the system given in Eq. (1) from given initial conditions to given final conditions while satisfying the limiting constraints

$$u(t) \in \mathcal{U}, \qquad \mathcal{U} = [-1, 1] \qquad \int_0^{t_f} |u(t)|\,\mathrm{d}t \le U \tag{5}$$

where t_f is the settling time. These constraints represent limited actuator authority [3] and limited fuel usage [23]. Further, the control is desired to be robust in terms of the vibration cancellation of characteristic modes [4,5,19], and stable in terms of keeping the excitations of the higher unmodeled modes below the acceptable values [14,27].

The robustness criteria can be considered in the design by including virtual flexible modes in the system dynamics given in Eq. (1) and appropriately computing the final states. These virtual poles can be either placed on top of the existing modes [3,4] or in the neighborhood of the modeled poles [28,29]. The resulting robustness

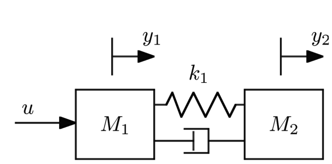

Fig. 1 Two mass example.

depends upon the location of these virtual poles [3,4,28,29]. After appending these virtual modes, the updated dynamics corresponding to the flexibility is given by $\{\boldsymbol{x_m} \in \mathbb{R}^m, \boldsymbol{A_m}, \boldsymbol{b_m}\}$, where the subscript m refers to the modeled poles $\{\omega \in \Omega_m, \zeta \in \mathbb{Z}_m\}$. The initial and final values of the states $\boldsymbol{x_m}$ are known from the problem definition.

Another method of specifying the robustness is to limit the residual vibrations corresponding to the virtual modes by terminal state inequality constraints [25,30]. The dynamics corresponding to these virtual modes represented by $\{\omega \in \Omega_{vm}, \zeta \in \mathbb{Z}_{vm}\}$ are given similar to Eq. (1) by $\{\boldsymbol{x_{vm}} \in \mathbb{R}^{\bar{n}}, \boldsymbol{A_{vm}}, \boldsymbol{b_{vm}}\}$. The states are known at the initial time and the values at the final time are constrained by

$$\boldsymbol{R}\,(\boldsymbol{x_{vm}}(t_f)) \le \boldsymbol{R}_{\max} \tag{6}$$

where $\boldsymbol{R}$: $\mathbb{R}^{\bar{n}} \to \mathbb{R}^{\bar{p}}$. The desired robustness is specified in terms of limiting the residual vibrations of these virtual modes given by

$$\boldsymbol{V}\,(\boldsymbol{x_{vm}}(t_f)) = (\boldsymbol{x_{vm}}(t_f) - \boldsymbol{x}_{\text{ref}})^T \boldsymbol{W} (\boldsymbol{x_{vm}}(t_f) - \boldsymbol{x}_{\text{ref}}) \tag{7}$$

which can be related to the vibration energy of the unmodeled flexible modes for suitable weight matrix $\boldsymbol{W}$ [7]. For the system given in Eq. (2),

$$\boldsymbol{W} = \begin{bmatrix} \left(\frac{1}{1-\zeta_1^2}\right) & -\left(\frac{\zeta_1}{1-\zeta_1^2}\right) \\ -\left(\frac{\zeta_1}{1-\zeta_1^2}\right) & \left(\frac{\zeta_1^2}{1-\zeta_1^2}\right) + \frac{1}{\omega_1^2} \end{bmatrix} \tag{8}$$

The constraints given in Eq. (6) are formulated for specifying the robustness or the "insensitivity" of the control to the variation in the characteristic mode that needs to be suppressed, and are therefore defined as the specified insensitivity constraints [25]. The limit $\boldsymbol{R}_{\max}$ is usually defined to allow not more than 10% of vibrations with respect to the unshaped step response [25]. Because the specified insensitivity constraints are defined in the neighborhood of the modes that need to be suppressed [25], additional switchings are usually required in the control to obtain the desired response.

The last design criterion is to prevent the excitations of the higher modes which are not included in the initial problem formulation because of mode truncation where a limit is usually set on the Bode gain such that all modes lying below that gain are neglected [3]. The fast switching control has been shown to overexcite some of the higher modes [8,27]. If the unmodeled plant modes are begin overexcited by the switched control, the excitations of these modes has to be considered in the final design by following an approach similar to that used for providing robustness [14]. Similar to the specified insensitivity constraints, band excitation constraints (BECs) can be defined that limit the overexcitations of the flexible modes in a user defined band [14]. These constraints are written as

$$\boldsymbol{E}\,(\boldsymbol{x_{hm}}(t_f)) \le \boldsymbol{E}_{\max} \tag{9}$$

where $\boldsymbol{x_{hm}}$ corresponds to higher mode states. The limit $\boldsymbol{E}_{\max}$ defines the acceptable excitation value and is usually specified to restrict the vibrations below 100–200% of the unshaped step response. As compared to the specified insensitivity, the BECs are important only for modes that are overexcited with respect to their unshaped response. On the other hand, the virtual modes used in the robustness definition are required to be suppressed below 5–10% of their unshaped response. Because these higher modes are not required to be suppressed, the desired response can usually be obtained by just modifying the switch timings instead of adding more control switchings as is usually required for insensitivity [14].

The states $\{\boldsymbol{x_{vm}}, \boldsymbol{x_{hm}}\}$ are appended and the dynamics corresponding to these constrained modes are represented by $\{\boldsymbol{x_{cm}} \in \mathbb{R}^n, \boldsymbol{A_{cm}}, \boldsymbol{b_{cm}}\}$. The states are known at the initial time, but the values at the final time t_f are known indirectly through the p terminal state inequalities represented as

$$\boldsymbol{F}\,(\boldsymbol{x_{cm}}(t_f)) \le \boldsymbol{F}_{\max} \tag{10}$$

where $\boldsymbol{F}$: $\mathbb{R}^n \to \mathbb{R}^p$.

The generalized vibration reduction design problem can now be formulated as a free-time optimal control problem given by

$$\begin{aligned} &\min \quad t_f \\ &\text{subject to } \dot{\boldsymbol{x}}_{\boldsymbol{r}}(t) = \boldsymbol{A_r}\boldsymbol{x_r}(t) + \boldsymbol{b_r}u(t) \\ &\qquad \boldsymbol{x_r}(0) = \boldsymbol{x_r}^{(0)}, \qquad \boldsymbol{x_r}(t_f) = \boldsymbol{x_r}^{(t_f)} \\ &\qquad \dot{\boldsymbol{x}}_{\boldsymbol{m}}(t) = \boldsymbol{A_m}\boldsymbol{x_m}(t) + \boldsymbol{b_m}u(t) \\ &\qquad \boldsymbol{x_m}(0) = \boldsymbol{x_m}^{(0)}, \qquad \boldsymbol{x_m}(t_f) = \boldsymbol{x_m}^{(t_f)} \\ &\qquad \dot{\boldsymbol{x}}_{\boldsymbol{cm}}(t) = \boldsymbol{A_{cm}}\boldsymbol{x_{cm}}(t) + \boldsymbol{b_{cm}}u(t) \\ &\qquad \boldsymbol{x_{cm}}(0) = \boldsymbol{x_{cm}}^{(0)} \\ &\qquad \dot{z}(t) = |u(t)|, \qquad z(0) = 0 \\ &\qquad \boldsymbol{F}(\boldsymbol{x_{cm}}(t_f)) \le \boldsymbol{F}_{\max}, \qquad z(t_f) \le U \\ &\qquad u(t) \in \mathcal{U}, \qquad \mathcal{U} = [-L, L] \end{aligned} \tag{11}$$

Another possible formulation represents the vibration reduction problem as a fixed-time optimal control problem where the terminal states are considered in a cost function in place of the inequality constraints. For a class of function definitions, the two formulations can be shown to be equivalent [15]. Although only the free-time formulations are considered in the current work, the analysis can be easily extended for the fixed-time minimum cost formulations.

A related problem is the design of vibration reduction shaping filters where the control is designed only for the flexible modes. For the rest-to-rest motion of flexible systems, the reference command is modified by convolving with either a single input shaping filter [30], or by a combination of transition shaping filters [6]. The analysis remains the same for such problems where only flexible mode dynamics are included in Eq. (11).

III. Necessary Conditions

The linear time-invariant system represented in Eq. (11) can be shown to be marginally stable and controllable [17]. The system is also normal as a result of the one dimensionality of the control space [21]. Hence a unique optimal solution is guaranteed to exist, and Pontryagin's minimum principle provides both the necessary as well as the sufficient conditions for optimality [16–18].

The Hamiltonian for the optimal control problem given in Eq. (11) can be written as [18,31,32]

$$\begin{aligned} \mathbf{H} = 1 &+ \boldsymbol{\lambda}^T(\boldsymbol{A_r}\boldsymbol{x_r} + \boldsymbol{b_r}u) + \boldsymbol{\mu}^T(\boldsymbol{A_m}\boldsymbol{x_m} + \boldsymbol{b_m}u) \\ &+ \boldsymbol{\eta}^T(\boldsymbol{A_{cm}}\boldsymbol{x_{cm}} + \boldsymbol{b_{cm}}u) + \nu|u| \end{aligned} \tag{12}$$

where $\boldsymbol{\lambda} \in \mathbb{R}^r$, $\boldsymbol{\mu} \in \mathbb{R}^m$, $\boldsymbol{\eta} \in \mathbb{R}^n$, and $\nu \in \mathbb{R}$ are the costate vectors. From the optimality conditions, the costate dynamics are given by

$$\begin{aligned} \dot{\boldsymbol{\lambda}}(t) &= -\mathbf{H}_{x_r}^T(t) = -\boldsymbol{A_r}^T\boldsymbol{\lambda}(t) \\ \dot{\boldsymbol{\mu}}(t) &= -\mathbf{H}_{x_m}^T(t) = -\boldsymbol{A_m}^T\boldsymbol{\mu}(t) \\ \dot{\boldsymbol{\eta}}(t) &= -\mathbf{H}_{x_{cm}}^T(t) = -\boldsymbol{A}_{\boldsymbol{cm}}^T\boldsymbol{\eta}(t) \qquad \dot{\nu}(t) = -\mathbf{H}_z^T(t) = 0 \end{aligned} \tag{13}$$

where the subscript represents the derivative. The Hamiltonian $\mathbf{H}$ is equal to zero for all time t since it is not an explicit function of time [16]. The transversality conditions for the terminal time t_f are [31,32]

$$\mathbf{H}\,(t_f) = 0, \qquad \boldsymbol{\eta}^T(t_f^-) = \boldsymbol{\alpha}^T\boldsymbol{F}_{x_{cm}}(\boldsymbol{x_{cm}}(t_f)), \qquad \nu(t_f^-) = \beta \tag{14}$$

where $\beta \in \mathbb{R}$ and $\boldsymbol{\alpha} \in \mathbb{R}^p$ are the Lagrange multipliers of the terminal state inequality constraints given in Eqs. (5) and (10), respectively. The multipliers $\boldsymbol{\alpha}$ and β satisfy the following KKT conditions [20]:

$$\begin{aligned} \boldsymbol{\alpha} \ge \mathbf{0}, \qquad \boldsymbol{\alpha} \circ (\boldsymbol{F}(\boldsymbol{x_{cm}}(t_f)) - \boldsymbol{F}_{\max}) = 0 \qquad \beta \ge \mathbf{0} \\ \beta(z(t_f) - U) = 0 \end{aligned} \tag{15}$$

where $\circ$ is the Hadamard vector product such that $(\boldsymbol{x} \circ \boldsymbol{y})_i = x_i y_i$.

The optimal solution is now obtained from the Pontryagin's minimum principle [16]

$$u^*(t) = \arg\min_{u\in\mathcal{U}} \mathbf{H}(\boldsymbol{x_r}, \boldsymbol{x_m}, \boldsymbol{x_{cm}}, z, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\eta}, \nu, u) \tag{16}$$

The resulting control is either bang–bang or bang–off–bang depending upon the limits specified for the fuel cost in Eq. (5). If the specified fuel constraint is inactive, the Hamiltonian becomes independent of the fuel cost as is clear from Eq. (15) that gives $\beta = 0$. As a result, the optimal control is bang–bang [33] and is given by the sign of the switching function. By denoting

$$\boldsymbol{B}^T = \begin{bmatrix} \boldsymbol{b}_r^T & \boldsymbol{b}_m^T & \boldsymbol{b}_{cm}^T \end{bmatrix} \qquad \boldsymbol{q}^*(t) = \begin{bmatrix} \boldsymbol{\lambda}^*(t) \\ \boldsymbol{\mu}^*(t) \\ \boldsymbol{\eta}^*(t) \end{bmatrix} \tag{17}$$

the optimal solution becomes

$$u^*(t) = \begin{cases} 1 & \boldsymbol{B}^T\boldsymbol{q}^*(t) < 0 \\ -1 & \boldsymbol{B}^T\boldsymbol{q}^*(t) > 0 \end{cases} \tag{18}$$

On the other hand, if the specified fuel constraint is active at optimality, the resulting control is bang–off–bang [23,24]. For such problems, the optimal control and the switching function are given by

$$u^*(t) = \begin{cases} 0 & \left|\frac{\boldsymbol{B}^T\boldsymbol{q}^*(t)}{\nu}\right| < 1 \\ -1 & \frac{\boldsymbol{B}^T\boldsymbol{q}^*(t)}{\nu} \geq 1 \\ 1 & \frac{\boldsymbol{B}^T\boldsymbol{q}^*(t)}{\nu} \leq -1 \end{cases} \tag{19}$$

From Eqs. (14) and (15), $\nu > 0$. Hence Eq. (19) can be modified to

$$u^*(t) = \begin{cases} 0 & -\nu < \boldsymbol{B}^T\boldsymbol{q}^*(t) < \nu \\ -1 & \boldsymbol{B}^T\boldsymbol{q}^*(t) \geq \nu \\ 1 & \boldsymbol{B}^T\boldsymbol{q}^*(t) \leq -\nu \end{cases} \tag{20}$$

IV. Parametric Optimization

The computation of optimal control from the Euler–Lagrange necessary conditions can be quite complicated and the solution is required to satisfy additional transversality and KKT conditions. For problems with a switched control as the optimal solution, an alternative parametric optimization approach exists that is computationally very efficient [1,19]. Such a switched control is expressed as [4]

$$U(s) = \frac{A_1 + A_2 e^{-t_1 s} + \cdots + A_{p+1} e^{-t_p s}}{s} \tag{21}$$

where $\boldsymbol{A}$ and $\boldsymbol{t}$ correspond to the amplitudes and the p switching times. The settling time is given by the last switch $t_f = t_p$. If the rigid-body modes are included in Eq. (11), the amplitudes for the bang–bang (bb), and bang–off–bang (bob) control are given as

$$\boldsymbol{A}_{bb} = [1 \quad -2 \quad 2 \quad \cdots \quad 1] \qquad \boldsymbol{A}_{bob} = [1 \quad -1 \quad \pm 1 \quad \cdots \quad 1] \tag{22}$$

If only flexible modes are considered in the filter design problem, the amplitudes can be written accordingly such that the dc gain of the resulting filter is unity [4,8]. The state response, terminal state constraints, and the actuator limitations are specified as analytical expressions of the switching times $\boldsymbol{t}$ that form the variable vector. The vibration energy of a flexible mode $\{\omega_i, \zeta_i\}$ with dynamics given in Eq. (4) is represented by the residual vibration which is given by [19]

$$\hat{V}_i(\omega_i, \zeta_i, \boldsymbol{t}) = \frac{e^{-\zeta_i\omega_i t_p}}{\omega_i^2\sqrt{1-\zeta_i^2}} \sqrt{(C(\omega_i, \zeta_i))^2 + (S(\omega_i, \zeta_i))^2} \tag{23}$$

where

$$\begin{aligned} C(\omega_i, \zeta_i, \boldsymbol{t}) &= A_1 + \sum_{j=1}^{p} A_{j+1} e^{\zeta_i\omega_i t_j} \cos\left(\omega_i\sqrt{1-\zeta_i^2}\, t_j\right) \\ S(\omega_i, \zeta_i, \boldsymbol{t}) &= \sum_{j=1}^{p} A_{j+1} e^{\zeta_i\omega_i t_j} \sin\left(\omega_i\sqrt{1-\zeta_i^2}\, t_j\right) \end{aligned} \tag{24}$$

A similar function is the vibration sensitivity function $\boldsymbol{V}$ that relates the residual vibrations corresponding to the filtered step command to the unshaped step input [25] and is given as

$$V_i = \left(\omega_i^2\sqrt{1-\zeta_i^2}\right)\hat{V}_i \tag{25}$$

For fixed endpoints corresponding to the modeled dynamics, the switched control given in Eq. (21) is required to cancel the modeled poles resulting in zero residual vibrations [3]. For the rest-to-rest motion of the rigid-body modes, the constraints are obtained by cancellation of rigid-body poles and application of final value theorem [9]. Constraints for robustness and flexible modes with higher-order repeated poles are likewise formulated [3,4].

The resulting parametric optimization formulation of Eq. (11) after including the respective constraints is now stated as

$$\begin{aligned} &\min_{\boldsymbol{t}} \quad t_p \\ &\text{subject to } \hat{\boldsymbol{V}}(\omega, \zeta, \boldsymbol{t}) = 0, \qquad \omega \in \Omega_m, \zeta_m \in \mathbb{Z}_m \\ &\qquad\qquad \hat{\boldsymbol{V}}(\bar{\omega}, \bar{\zeta}, \boldsymbol{t}) \leq \boldsymbol{V}_{\max}, \qquad \zeta \in \Omega_{cm}, \bar{\zeta} \in \mathbb{Z}_{cm} \\ &\sum_{j=1}^{p} A_{j+1} t_j = 0 \qquad \sum_{j=1}^{p} A_{j+1} t_j^2 - 2y_{\text{ref}} = 0 \\ &A_1 t_1 + \sum_{j=2}^{p}\left|\sum_{k=1}^{j} A_k\right|(t_j - t_{j-1}) \leq U \end{aligned} \tag{26}$$

Here y_{ref} represents the reference point for the rest-to-rest motion of the rigid-body modes. The solution can be obtained by solving Eq. (26) using nonlinear optimization routines like SNOPT [34] and MATLAB.

For the formulation given in Eq. (26), the parametric solution is allowed to have a variable number of switches. Although the rigid-body constraints and the fuel usage constraints are convex, the residual vibration function is not convex in the switch variable $\boldsymbol{t}$. As a result, only the local optimality of the solution can be guaranteed by the necessary conditions given by KKT [20,35]. These include the constraints given in Eq. (26) and the following conditions:

$$\begin{aligned} &\boldsymbol{\psi} \circ (\hat{\boldsymbol{V}}(\bar{\omega}, \bar{\zeta}, \boldsymbol{t}) - \boldsymbol{V}_{\max}) = \boldsymbol{0}, \qquad \bar{\omega} \in \Omega_{cm}, \qquad \bar{\zeta} \in \mathbb{Z}_{cm} \\ &\varrho\left(A_1 t_1 + \sum_{j=2}^{p}\left|\sum_{k=1}^{j} A_k\right|(t_j - t_{j-1}) - U\right) = 0 \\ &\boldsymbol{\psi} \geq \boldsymbol{0}, \qquad \varrho \geq 0 \\ &0 + \boldsymbol{\xi}^T\hat{\boldsymbol{V}}_{\boldsymbol{t}_j}(\omega, \zeta, \boldsymbol{t}) + \sigma_1(A_{j+1}) + \sigma_2(2A_{j+1}t_j) \\ &\quad + \boldsymbol{\psi}^T\hat{\boldsymbol{V}}_{\boldsymbol{t}_j}(\bar{\omega}, \bar{\zeta}, \boldsymbol{t}) + \varrho\left(\left|\sum_{k=1}^{j} A_k\right| - \left|\sum_{k=1}^{j+1} A_k\right|\right) = 0 \\ &\forall\ j = 1, \ldots, p-1, \qquad \omega \in \Omega_m, \qquad \zeta \in \mathbb{Z}_m \\ &1 + \boldsymbol{\xi}^T\hat{\boldsymbol{V}}_{\boldsymbol{t}_p}(\omega, \zeta, \boldsymbol{t}) + \sigma_1(A_{p+1}) + \sigma_2(2A_{p+1}t_p) \\ &\quad + \boldsymbol{\psi}^T\hat{\boldsymbol{V}}_{\boldsymbol{t}_p}(\bar{\omega}, \bar{\zeta}, \boldsymbol{t}) + \varrho\left(\left|\sum_{k=1}^{p} A_k\right|\right) = 0 \end{aligned} \tag{27}$$

where $\boldsymbol{\psi} \in \mathbb{R}^n$, $\varrho \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^m$, and $\boldsymbol{\sigma} \in \mathbb{R}^2$ are the Lagrange multipliers, and $\circ$ is the Hadamard vector product. The vector $\hat{\boldsymbol{V}}_{\boldsymbol{t}_j}$ represents the derivative of $\hat{\boldsymbol{V}}$ with respect to t_j. An algorithm is presented next for verifying the global optimality of the parametric solution.

V. Parametric Solution Verification

Sufficient conditions for the optimality of the switched control is provided by Pontryagin's minimum principal and can be used for the verification of the parametric solution obtained by Eq. (26). The approach presented in [21] is now extended for general vibration reduction problems with terminal state inequalities.

From Eq. (13), the costate vectors are given by

$$\begin{aligned} \boldsymbol{\lambda}(t) = \boldsymbol{e}^{-\boldsymbol{A}_r^T t}\boldsymbol{\lambda}(0) \qquad \boldsymbol{\mu}(t) = \boldsymbol{e}^{-\boldsymbol{A}_m^T t}\boldsymbol{\mu}(0) \\ \boldsymbol{\eta}(t) = \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t}\boldsymbol{\eta}(0) \qquad \nu(t) = \nu(0) \end{aligned} \tag{28}$$

Following the representation given in Eq. (17), for bang–bang control, the switching function shown in Eq. (18) becomes

$$\begin{bmatrix} \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t} \end{bmatrix} \boldsymbol{q}^*(0) = 0 \tag{29}$$

for $t = t_1, t_2, \ldots, t_{p-1}$. The switching time $\boldsymbol{t}$ computed by parametric optimization routine must therefore satisfy

$$\boldsymbol{S}_{bb}\boldsymbol{q}^*(0) = \boldsymbol{0} \tag{30}$$

where $\boldsymbol{S}_{bb} \in \mathbb{R}^{(p-1)\times(r+m+n)}$ is given by

$$\boldsymbol{S}_{bb} = \begin{bmatrix} \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t_1} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t_1} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t_1} \\ \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t_2} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t_2} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t_2} \\ \vdots & \vdots & \vdots \\ \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t_{p-1}} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t_{p-1}} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t_{p-1}} \end{bmatrix} \tag{31}$$

Hence the initial values of the costate vectors can be found by the null space of the matrix $\boldsymbol{S}_{bb}$ which can then be scaled for satisfying the Hamiltonian requirement that $\mathbf{H}(0) = 0$ written as

$$\begin{bmatrix} \boldsymbol{b}_r^T & \boldsymbol{b}_m^T & \boldsymbol{b}_{cm}^T \end{bmatrix} \boldsymbol{q}^*(0) = -1 \tag{32}$$

from Eq. (12) for zero initial conditions. This vector can then be used in Eq. (28) to verify that the profile obtained from Eq. (18) matches with the parametric solution.

The dimensions of the state vectors and the number of switchings in the optimal solution determine the dimension of the null space for $\boldsymbol{S}_{bb}$. Because any vector in the null space can be scaled according to $\mathbf{H}(0) = 0$, more conditions are required other than Eqs. (30) and (32) to make this approach practical. This becomes more critical for problems with terminal state inequalities because a large number of unmodeled modes is usually considered in the problem which increases the dimension of the null space. The uniqueness of the null space solution is a matter of chance and not a property of vibration reduction problem formulations, as incorrectly stated in [22]. This can be seen from the infinite zeros of the switched control given in Eq. (21) where the same number of switchings cancels multiple system poles if located on any of these zeros. For such a case the dimension of the null space increases with an increase in the number of system poles and is not unity.

Additional equations are obtained from the transversality conditions given in Eqs. (14) and (15) that can be written in terms of the initial values of the costate vectors as

$$\begin{aligned} \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t_p}\boldsymbol{\eta}^*(0) - \boldsymbol{V}_{\boldsymbol{x}_{cm}}^T(\boldsymbol{x}_{cm}(t_p))\boldsymbol{\alpha}^* = 0 \qquad \nu^* = \beta \qquad \boldsymbol{\alpha}^* \geq \boldsymbol{0} \\ (\boldsymbol{V}(\boldsymbol{x}_{cm}(t_p)) - \boldsymbol{V}_{\max}) \circ \boldsymbol{\alpha}^* = 0 \qquad \beta^* \geq \boldsymbol{0} \\ \beta^*(z(t_p) - U) = 0 \end{aligned} \tag{33}$$

where $\boldsymbol{\alpha} \in \mathbb{R}^n$ and β are the additional Lagrange multipliers, and $\circ$ is the Hadamard vector product. The vector $\boldsymbol{V}_{\boldsymbol{x}_{cm}}$ is obtained by taking the derivative of Eq. (7) with respect to the states $\boldsymbol{x}_{cm}$, and the states $\boldsymbol{x}_{cm}(t_p)$ and $z(t_p)$ are computed using the parametric solution in the state dynamics.

Inclusion of Eq. (33) reduces the dimensions of the null space. However, the uniqueness of the null space solution still cannot be guaranteed. As a result multiple feasible solutions of the null space vector might be required to be compared to the switching profile. If the values of the switching function at times other than the switching times are also used, the comparison process can be included directly in the computation process. In this approach, the times between two switches are discretized and inequality constraints are included in the computation corresponding to the sign of the switching function at these sample times. The time bands are discretized using Δt as the sampling time to get a known finite number of inequality conditions. For the switching function given in Eq. (18)

$$\begin{bmatrix} \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t} \end{bmatrix} \boldsymbol{q}^*(0) \leq 0 \tag{34}$$

for $t \in \mathbb{T}_1$ and

$$\begin{bmatrix} \boldsymbol{b}_r^T \boldsymbol{e}^{-\boldsymbol{A}_r^T t} & \boldsymbol{b}_m^T \boldsymbol{e}^{-\boldsymbol{A}_m^T t} & \boldsymbol{b}_{cm}^T \boldsymbol{e}^{-\boldsymbol{A}_{cm}^T t} \end{bmatrix} \boldsymbol{q}^*(0) \geq 0 \tag{35}$$

for $t \in \mathbb{T}_2$ where

$$\begin{aligned} \mathbb{T}_1 &= (0, \Delta t, \ldots, t_1) \cup (t_2, t_2 + \Delta t, \ldots, t_3) \\ &\quad \cup (t_4, t_2 + \Delta t, \ldots, t_5) \cup \ldots \\ \mathbb{T}_2 &= (t_1, \Delta t, \ldots, t_2) \cup (t_3, t_2 + \Delta t, \ldots, t_4) \\ &\quad \cup (t_5, t_2 + \Delta t, \ldots, t_6) \cup \ldots \end{aligned} \tag{36}$$

denotes the time bands when the control is 1 and -1, respectively. The inequalities given in Eqs. (34) and (35) are represented as

$$\boldsymbol{P}_{bb}\boldsymbol{q}^*(0) \leq 0 \quad \text{and} \quad \boldsymbol{N}_{bb}\boldsymbol{q}^*(0) \geq 0 \tag{37}$$

respectively. The matrices $\boldsymbol{P}_{bb} \in \mathbb{R}^{\tau_1\times(r+m+n)}$ and $\boldsymbol{N}_{bb} \in \mathbb{R}^{\tau_2\times(r+m+n)}$ where τ_1 and τ_2 are the cardinality of the sets $\mathbb{T}_1$ and $\mathbb{T}_2$, respectively. If these conditions are included in the computation of costate vectors, the parametric solution can be directly verified for optimality.

An upper limit on the value of the time step used for sampling the time bands can be obtained from the dynamics of the switching function

$$\mathbb{S}(t) = \boldsymbol{b}_r^T\boldsymbol{\lambda}^*(t) + \boldsymbol{b}_m^T\boldsymbol{\mu}^*(t) + \boldsymbol{b}_{cm}^T\boldsymbol{\eta}^*(t) \tag{38}$$

The corresponding costate dynamics are given by Eq. (13). Because the switching function is linear in costate vectors, the highest frequency component is given by the frequency of the highest flexible mode ω_h used in the definition of $\{\boldsymbol{x}_m, \boldsymbol{x}_{cm}\}$. The sample time is required to capture at least one point for the positive and negative regions in the wave response of the mode ω_h. Hence the upper limit is obtained as

$$\Delta t \leq \frac{\pi}{2\omega_h} \tag{39}$$

Because the equality and inequality constraints given in Eqs. (30) and (32–35) are convex in variables $\boldsymbol{\lambda}^*(0)$, $\boldsymbol{\mu}^*(0)$, $\boldsymbol{\eta}^*(0)$, $\nu^*(0)$, $\boldsymbol{\alpha}^*$, and β^*, the resulting problem can be formulated as testing the feasibility of a LP. Denoting the variable vectors as

$$\boldsymbol{y}^* = \begin{bmatrix} \boldsymbol{\lambda}^*(0) \\ \boldsymbol{\mu}^*(0) \\ \boldsymbol{\eta}^*(0) \\ \nu^* \\ \boldsymbol{\alpha}^* \\ \beta^* \end{bmatrix} \tag{40}$$

the resulting LP is

$$
\begin{aligned}
\begin{bmatrix} S_{bb} & 0 & \mathbf{0} & 0 \end{bmatrix} y^* &= 0 \\
\begin{bmatrix} b_r^T & b_m^T & b_{cm}^T & 0 & \mathbf{0} & 0 \end{bmatrix} y^* &= -1 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & e^{-A_{cm}^T t_p} & 0 & -V_{x_{cm}}^T(t_p) & 0 \end{bmatrix} y^* &= 0 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & \mathbf{0} & 1 \end{bmatrix} y^* &= 0 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \operatorname{diag}(V(t_p) - V_{\max}) & 0 \end{bmatrix} y^* &= 0 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & (z(t_p) - U) \end{bmatrix} y^* &= 0 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & -1 \end{bmatrix} y^* &\le 0 \\
\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & -I & 0 \end{bmatrix} y^* &\le 0 \\
\begin{bmatrix} P_{bb} & 0 & 0 & 0 \\ -N_{bb} & 0 & 0 & 0 \end{bmatrix} y^* &\le 0
\end{aligned}
\tag{41}
$$

Because the LP is a convex problem, Eq. (41) can be solved very efficiently using interior point methods [26]. The existence of a feasible solution of Eq. (41) directly provides the sufficient conditions for the verification of optimality of the parametric bang–bang solution computed by Eq. (26).

Similar analysis can be given for the bang–off–bang control. From Eq. (20) the switching condition is

$$
\begin{bmatrix} b_r^T e^{-A_r^T t} & b_m^T e^{-A_m^T t} & b_{cm}^T e^{-A_{cm}^T t} \end{bmatrix} q^*(0) = \mp \nu^* \tag{42}
$$

for $t = t_1, t_2, \ldots, t_{p-1}$ where the sign of the coefficient of ν^* depends upon the nature of the switch transitions, and is +ve for $0 \to 1$ and $1 \to 0$ transitions, and −ve for $0 \to -1$ and $-1 \to 0$ transitions. The switching time $\boldsymbol{t}$ computed by the parametric optimization routine must therefore satisfy

$$
\begin{bmatrix} S_{bob} & \pm 1 & \mathbf{0} & 0 \end{bmatrix} q^* = \mathbf{0} \tag{43}
$$

As before, the time zones are discretized to produce a finite number of inequalities satisfying the switching conditions specified in Eq. (20). The time zones are divided into three sets $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3\}$ corresponding to the times where control u is 1, −1, and 0, respectively. The inequalities are obtained as

$$
\begin{bmatrix} -P_{bob} & -1 \\ N_{bob} & -1 \\ Z_{bob} & 1 \\ -Z_{bob} & 1 \end{bmatrix} \begin{bmatrix} \lambda^*(0) \\ \mu^*(0) \\ \eta^*(0) \\ \nu^* \end{bmatrix} \le 0 \tag{44}
$$

where the matrices $P_{bob} \in \mathbb{R}^{\tau_1 \times (r+m+n)}$, $N_{bob} \in \mathbb{R}^{\tau_2 \times (r+m+n)}$, and $Z_{bob} \in \mathbb{R}^{\tau_3 \times (r+m+n)}$ where τ_1, τ_2, and τ_3 are the cardinality of the sets $\mathbb{T}_1$, $\mathbb{T}_2$, and $\mathbb{T}_3$, respectively. These inequalities and the switching conditions can be combined with other transversality and KKT conditions to get another LP that provides a direct verification of bang–off–bang profiles.

VI. Numerical Examples

An example is now considered with terminal state vibration sensitivity constraints and a specified fuel cost. The problem is the design of a time optimal specified excitation (TOSE) filter for an undamped flexible system [14]. The system is assumed to have a single nominal mode at $\omega_m = 1$ and $\zeta_m = 0$. The BECs are given for two frequency bands at $\Omega_{um_1} = [2, 3]$ and $\Omega_{um_2} = [6, 7]$ for limiting the vibration sensitivities below the allowable values. These constraints are similar to the specified insensitivity constraints [25] used for making the control more robust. The resulting time optimal control formulation becomes

$$
\begin{aligned}
\min \quad & \int_0^{t_f} \mathrm{d}t \\
\text{subject to } & \dot{\bar{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
& \bar{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \\
& \bar{x}(t_f) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \\
& \max_{2.0 \le \omega \le 3.0} V(\omega) \le 1.2 \\
& \max_{6.0 \le \omega \le 7.0} V(\omega) \le 1.5 \\
& \int_0^{t_f} |u(t)| \, \mathrm{d}t \le U \\
& -1 \le u(t) \le 1
\end{aligned}
\tag{45}
$$

where $V(\omega)$ represents the vibration sensitivity. The specified fuel cost limit is chosen such that the optimal solution is a bang–bang control for one case, and a bang–off-bang control for the other. The solutions are obtained by parametric optimization formulations and are verified for optimality by solving the respective LP. The nonlinear parameter optimization is solved using SNOPT [34] and MOSEK [36] is used for solving the LP.

For the bang–bang case, the fuel cost constraint is redundant. The resulting 8-switch bang–bang optimal control and the system response is shown in Fig. 2. The parametric solution is verified by the proposed LP approach, and the Hamiltonian is shown in Fig. 3. The switching, indeed, occurs according to the sign of the $b^T\lambda$ function. Because Eq. (45) is a minimum time problem, the Hamiltonian is zero at all times [16].

For the bang–off–bang case, the fuel cost constraint becomes active. The parametric solution is computed to be a 6-switch bang–off–bang control. The LP approach is again used to verify the optimality of the solution. The resulting response and the Hamiltonian are shown in Figs. 4 and 5. The switchings can be seen to match the profile predicted by the switching conditions given in Eq. (20).

VII. Discrete Time Formulation

The computation of optimal control for vibration reduction problems is a multistep process. First the problem is formulated as an optimal control problem, the necessary conditions are then written, and the optimal solution is predicted based on the switching function. A parametric optimization problem is then formulated to obtain the switched control. This solution is then verified for optimality using the LP based approach.

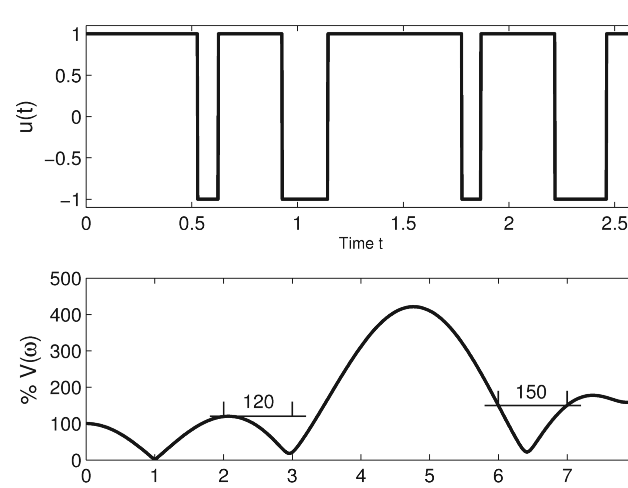

Fig. 2 System response $y(t)$, control input $u(t)$, and the vibration sensitivity for problem (45) with the bang–bang solution. The settling time is 2.46 s.

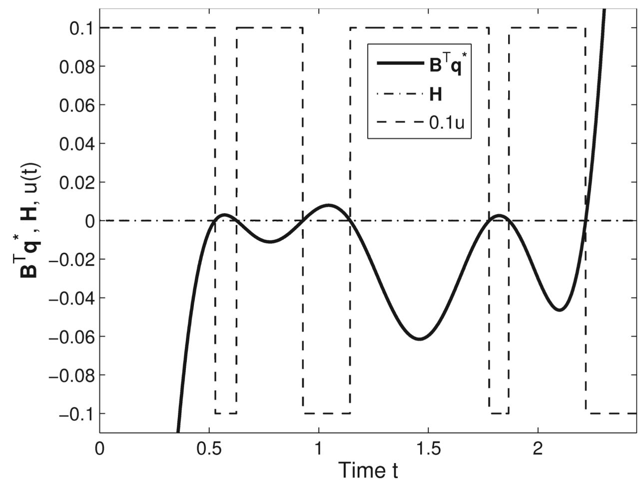

Fig. 3 Switching function, Hamiltonian, and scaled control input for problem (45) with the bang–bang solution.

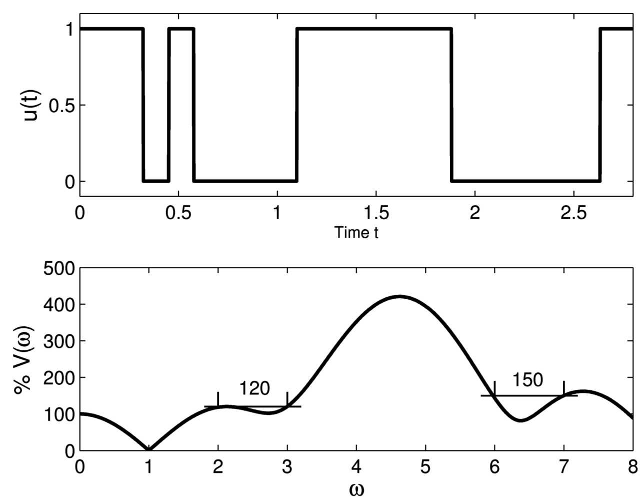

Fig. 4 System response $y(t)$, control input $u(t)$, and the vibration sensitivity for problem (45) with the bang–off–bang solution. The settling time is 2.63 s for $U = 1.23$. Only the fuel cost in making the $0 \to 1$ transition is considered.

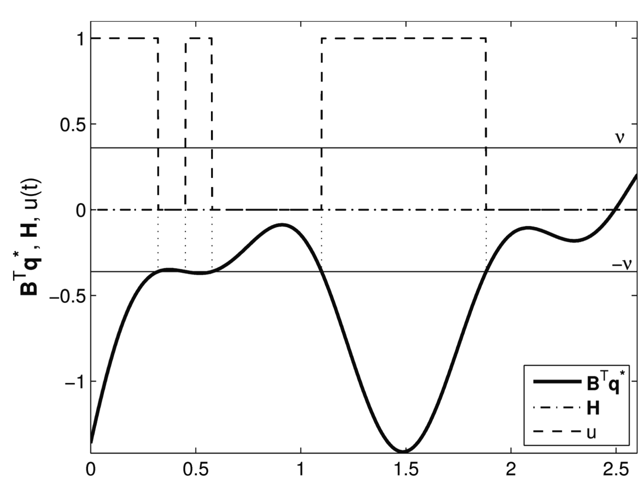

Fig. 5 Switching function, Hamiltonian, and control input for problem (45) with the bang–off–bang solution. ν is the Lagrange multiplier of specified fuel constraint. Switching times are projected on the line corresponding to $-\nu$ by dotted lines.

An approximate optimal solution can be obtained by formulating the problem in the discrete domain. Because the resulting problem is convex, the KKT conditions are both necessary as well as sufficient [20]. The optimal solution can be easily computed and directly implemented on digital controllers. For the problem given in Eq. (11), the discrete equivalent formulation can be obtained as

$$
\begin{aligned}
\min \quad & N \\
\text{subject to } & \hat{\boldsymbol{x}}_r(h+1) = \hat{\boldsymbol{A}}_r \hat{\boldsymbol{x}}_r(h) + \hat{\boldsymbol{b}}_r u(h) \\
& \hat{\boldsymbol{x}}_r(0) = \boldsymbol{x}_r^{(0)}, \qquad \hat{\boldsymbol{x}}_r(N) = \boldsymbol{x}_r^{(t_f)} \\
& \hat{\boldsymbol{x}}_m(h+1) = \hat{A}_m \hat{\boldsymbol{x}}_m(h) + \hat{\boldsymbol{b}}_m u(h) \\
& \hat{\boldsymbol{x}}_m(0) = \boldsymbol{x}_m^{(0)}, \qquad \hat{\boldsymbol{x}}_m(N) = \boldsymbol{x}_m^{(t_f)} \\
& \hat{\boldsymbol{x}}_{cm}(h+1) = \hat{\boldsymbol{A}}_{cm} \hat{\boldsymbol{x}}_{cm}(h) + \hat{\boldsymbol{b}}_{cm} u(h) \\
& \hat{\boldsymbol{x}}_{cm}(0) = \boldsymbol{x}_{cm}^{(0)} \\
& \sum_{h=0}^{N} |u(h)| T_s \leq U \\
& \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}_{cm}(N)) \leq \boldsymbol{F}_{\max} \\
& u(h) \in \mathcal{U}, \qquad \mathcal{U} = [-L, L]
\end{aligned}
\tag{46}
$$

where $\hat{\cdot}$ represents the discretized states and the discrete equivalents of the constraint functions. T_s is the sample time, and NT_s is the final settling time. This convex problem directly provides an approximate optimal solution of the original problem given in Eq. (11).

VIII. Conclusions

A new approach is presented for the verification of optimality of the switched control computed by a parametric optimization technique. A constrained time optimal control problem is formulated for the vibration reduction in flexible structures where the terminal states are limited by inequality constraints. Pontryagin's minimum principle and the switching function are used to obtain the conditions for verification of the parametric optimization solution. The verification problem is shown to be a linear program that can be solved very efficiently in polynomial time. A new time sampling based algorithm is derived that provides a direct test for the verification. The proposed approach is demonstrated in example problems with bang–bang and bang–off–bang control solutions.

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